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Decoherence of continuous variable quantum information in non-Markovian channels

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Abstract

Decoherence of quantum information carried by single-mode and two-mode Gaussian states is investigated under the influence of non-Markovian quantum channels, where correlation times of reservoir variables are assumed to take finite values. Degradation of purity, distinguishability and non-classicality of single-mode Gaussian states and relaxation of entanglement of two-mode Gaussian states are examined. The results show the importance of the non-Markovian effect of a thermal reservoir. Furthermore, continuous variable quantum teleportation is considered when a sender and a receiver share a two-mode Gaussian state through non-Markovian quantum channels.

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1. Introduction

There have been considerable advances in the field of the quantum information science [1, 2]. Among others, quantum information processing is essential in quantum cryptography, quantum communication and quantum computation. These are important not only in information technology, but also in the basic principles of quantum mechanics. Quantum states that carry information, however, are quite fragile under the influence of a thermal reservoir (an external environmental system) which causes decoherence of quantum information. When quantum information processing is performed in the real world, the quantum systems inevitably suffer losses of purity, distinguishability, non-classicality, entanglement and so on. Decoherence of quantum states is investigated by means of the several methods ranging from the phenomenological theory to the microscopic theory. In the phenomenological treatments [3–7], the loss of coherence is characterized by the several energy and phase relaxation times. These phenomenological methods are extended to treat the dynamical effects of a thermal reservoir in terms of stochastic processes, typical examples of which are the Gauss–Markov process and the two-state jump Markov process [8–12]. In the microscopic approach, the environment itself is considered to be a quantum system with quantum mechanical reservoir variables which are to be eliminated, for example, by means of the projection operator method [13–16] or by the path integral method [17]. Although in many cases, the Markovian approximation is assumed when the reservoir variables are eliminated, it has recently been shown that the non-Markovian effect plays an important role in the relaxation process of quantum information [18–24]. In particular, when a high-speed quantum information processing is required, the non-Markovian effect becomes essential since the characteristic time of dynamics of the relevant system may be comparable with the reservoir correlation time.

The present author has recently considered the non-equilibrium dynamics of quantum information of qubits by means of the phenomenological method [7], the stochastic method [25] and the microscopic method [26]. In particular, the microscopic theory shows that the non-Markovian effect is very important in quantum information processing of qubits. Thus, in this paper, we extend the non-Markovian theory of decoherence [26] so that it can treat quantum information processing of continuous variables. The continuous variable quantum information processing that uses quantum optical phenomena can implement quantum cryptography [27], quantum teleportation [28, 29], quantum dense coding [30, 31], entanglement swapping [32] and quantum computation [33]. The effects of an external environmental system on the continuous variable quantum information processing have been investigated in detail within the Markovian approximation [34–36]. Therefore, we will investigate decoherence of continuous variable quantum information under the influence of a non-Markovian thermal reservoir, using the microscopic approach which assumes a system–reservoir interaction and applies the projector operator method to eliminate the reservoir variables with finite correlation time. This paper is organized as follows. In section 2, using the projection operator method, we derive a non-Markovian quantum channel for bosonic systems and we briefly summarize the basic properties. In section 3, we investigate decay of purity, distinguishability and non-classicality of single-mode Gaussian states caused by the non-Markovian quantum channel. In section 4, we obtain the entanglement of formation of symmetric two-mode Gaussian states in the non-Markovian quantum channel to evaluate decoherence of entanglement. The results show that the non-Markovian effect suppresses the decoherence of quantum information. In section 5, we investigate the condition that the non-Markovian quantum channel becomes an entanglement-breaking channel. In section 6, we consider the quantum teleportation of continuous variables that is performed under the influence of the non-Markovian thermal reservoir and calculate the fidelity to show how faithfully a Gaussian state is teleported. We give concluding remarks in section 7.

2. The non-Markovian quantum channel for bosonic systems

In this section, making use of the projection operator method [13–15] that yields a time-convolutionless quantum master equation of a reduced density operator, we derive a non-Markovian quantum channel for bosonic systems that is used for investigating decoherence in information processing of continuous variables. A quantum state $\hat{W}(t)$ of a relevant system and thermal reservoir is subject to the Liouville–von Neumann equation

$$\frac{\partial}{\partial t}\hat{W}(t) = -\frac{\mathrm{i}}{\hbar}[\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{R}} + \hat{H}_{\mathrm{SR}}, \hat{W}(t)], \tag{1}$$

where $\hat{H}_{\mathrm{S}} = \hbar\omega\hat{a}^{\dagger}\hat{a}$ and $\hat{H}_{\mathrm{R}}$ are the Hamiltonians of the relevant system and thermal reservoir, respectively, and $\hat{H}_{\mathrm{SR}}$ is the interaction Hamiltonian between them, where $\hat{a}$ and $\hat{a}^{\dagger}$ are bosonic annihilation and creation operators of the relevant system, respectively. We assume that the interaction Hamiltonian $\hat{H}_{\mathrm{SR}}$ is given by

$$\hat{H}_{\mathrm{SR}} = \hbar\lambda(\hat{a}^\dagger \hat{R} + \hat{a}\hat{\mathrm{R}}^\dagger), \tag{2}$$

where $\hat{R}$ represents some operator of the thermal reservoir and the parameter λ is the coupling constant. The system–reservoir interaction Hamiltonian $\hat{H}_{\mathrm{SR}}$ is convenient to make clear the non-Markovian effects on the decoherence of quantum information since the derived non-Markovian master equation becomes identical with the well-known Markovian master equation of the Lindblad form when the correlation time of the thermal reservoir is sufficiently short. However, the interaction Hamiltonian $\hat{H}_{\mathrm{SR}}$ is not unique. The non-Markovian effects obtained by the exactly solvable model in which the system–reservoir interaction Hamiltonian has the form $\hat{H}_{\mathrm{SR}} = \lambda(\hat{a} + \hat{a}^\dagger)(\hat{R} + \hat{R}^\dagger)$ has been investigated in detail [37, 38]. Moreover, the difference approaches to the non-Markovian relaxation have been given [40, 41].

When we apply the projection operator method to eliminate the reservoir variables from equation (1), we can obtain the time-convolutionless quantum master equation of the relevant system up to the second order (the non-trivial lowest order) with respect to the system–reservoir coupling constant λ

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho}(t) = -\mathrm{i}\omega[\hat{a}^\dagger\hat{a}, \hat{\rho}(t)] + \phi^*_{+-}(t)[\hat{a}^\dagger, \hat{\rho}(t)\hat{a}] + \phi_{+-}(t)[\hat{a}^\dagger\hat{\rho}(t), \hat{a}] \\ + \phi^*_{-+}(t)[\hat{a}, \hat{\rho}(t)\hat{a}^\dagger] + \phi_{-+}(t)[\hat{a}\hat{\rho}(t), \hat{a}^\dagger],\end{aligned} \tag{3}$$

where $\hat{\rho}(t) = \mathrm{Tr}_{\mathrm{R}}\, \hat{W}(t)$ is the reduced density operator of the relevant system and Tr_{R} indicates the trace operation over the Hilbert space of the thermal reservoir. The functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ which determine the relaxation process are given by

$$\phi_{+-}(t) = \lambda^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle \hat{R}^\dagger(\tau)\hat{R}(0)\rangle_{\mathrm{R}} \tag{4}$$

$$\phi_{-+}(t) = \lambda^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} \langle \hat{R}(\tau)\hat{R}^\dagger(0)\rangle_{\mathrm{R}}, \tag{5}$$

where $\langle\cdots\rangle_{\mathrm{R}}$ stands for the average value in the equilibrium state of the thermal reservoir. In the Markovian limit (or equivalently the narrowing limit), the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ are replaced with $\phi_{+-}(\infty)$ and $\phi_{-+}(\infty)$, respectively, in equation (3). The imaginary parts of the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ contribute to the frequency shift of the relevant system and they are irrelevant to the decoherence. Hence in the rest of this paper, we may assume that the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ are real and we may ignore the unitary part of the time-evolution.

To solve the time-convolutionless quantum master equation (3), we introduce the s-ordered quasi-probability distribution $F_s(t;z)$ [39, 42] which represents the quantum state $\hat{\rho}(t)$ of the relevant system,

$$F_s(t;z) = \mathrm{Tr}\,[\hat{T}(z;-s)\hat{\rho}] \qquad \hat{\rho}(t) = \int \frac{\mathrm{d}^2 z}{\pi} F_s(t;z)\hat{T}(z;s) \tag{6}$$

with

$$\hat{T}(z;s) = \int \frac{\mathrm{d}^2\alpha}{\pi} \hat{D}(\alpha)\, \mathrm{e}^{\frac{1}{2}s|\alpha|^2 - \alpha z^* + \alpha^* z}, \tag{7}$$

where $\hat{D}(\alpha) = \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ is the displacement operator. In particular, the functions $F_{-1}(t;z)$, $F_0(t;z)$ and $F_1(t;z)$ are the Glauber–Sudarshan P-function, the Wigner function and the

Husimi–Kano Q-function. The s-ordered quasi-probability distribution $F_s(t;z)$ has the property

$$\langle\{\hat{a}^{\dagger m}\hat{a}^n\}_s\rangle_t = \mathrm{Tr}[\{\hat{a}^{\dagger m}\hat{a}^n\}_s\hat{\rho}(t)] = \int \frac{\mathrm{d}^2 z}{\pi} z^{*m} z^n F_s(t;z), \tag{8}$$

where the s-ordered product of annihilation and creation operators is defined by

$$\{\hat{a}^{\dagger m}\hat{a}^n\}_s = \left.\frac{\partial^{m+n}}{\partial z^m \partial(-z^*)^n}\,\mathrm{e}^{z\hat{a}^\dagger - z^*\hat{a} + \frac{1}{2}s|z|^2}\right|_{z=0}. \tag{9}$$

When the quantum state $\hat{\rho}(t)$ obeys the quantum master equation (3), the quasi-probability distribution $F_s(t;z)$ is subject to the differential equation

$$\begin{aligned}\frac{\partial}{\partial t}F_s(t;z) &= [\phi_{-+}(t) - \phi_{+-}(t)]\left(\frac{\partial}{\partial z}z + \frac{\partial}{\partial z^*}z^*\right)F_s(t;z)\\ &\quad + [(1+s)\phi_{-+}(t) + (1-s)\phi_{+-}(t)]\left(\frac{\partial^2}{\partial z\partial z^*}\right)F_s(t;z)\end{aligned} \tag{10}$$

the solution of which is given by

$$F_s(t;z) = \int \frac{\mathrm{d}^2 z'}{\pi} G_s(t;z|t';z') F_s(t';z'), \tag{11}$$

where the Green function $G_s(t;z|t';z')$ is

$$G_s(t;z|t';z') = \frac{1}{\Delta_s(t,t')}\exp\left[-\frac{|z - z'\,\mathrm{e}^{-\Gamma(t,t')}|^2}{\Delta_s(t,t')}\right] \tag{12}$$

with

$$\Gamma(t,t') = \int_{t'}^{t} \mathrm{d}\tau[\phi_{-+}(\tau) - \phi_{+-}(\tau)] \tag{13}$$

$$\Delta_s(t,t') = \int_{t'}^{t} \mathrm{d}\tau[(1+s)\phi_{-+}(\tau) + (1-s)\phi_{+-}(\tau)]\,\mathrm{e}^{-2\Gamma(t,\tau)}. \tag{14}$$

Equations (11)–(14) are derived in the appendix. Equations (11) and (12) define the non-Markovian quantum channel in the phase space. Since it transforms a Gaussian state into another Gaussian state, it is called the Gaussian channel.

To proceed further, we must determine the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ given by equations (4)–(5). We assume here that the correlation functions of the reservoir variables decay exponentially with time t [26]

$$\lambda^2\langle\hat{R}^\dagger(t)\hat{R}(0)\rangle_R = \frac{1}{\tau_R}G_{+-}\,\mathrm{e}^{\mathrm{i}\omega' t - t/\tau_R} \tag{15}$$

$$\lambda^2\langle\hat{R}(t)\hat{R}^\dagger(0)\rangle_R = \frac{1}{\tau_R}G_{-+}\,\mathrm{e}^{-\mathrm{i}\omega' t - t/\tau_R}, \tag{16}$$

where $G_{+-} = G_{-+}^* = \lambda^2\langle\hat{R}^\dagger(0)\hat{R}(0)\rangle_R$ and ω' is some frequency of the reservoir variable and τ_R is the reservoir correlation time. Then the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ are expressed as

$$\phi_{+-}(t) = \phi_{+-}(\infty)(1 - \mathrm{e}^{-t/\tau_R - \mathrm{i}(\omega-\omega')t}) \tag{17}$$

$$\phi_{-+}(t) = \phi_{--}(\infty)(1 - \mathrm{e}^{-t/\tau_R + \mathrm{i}(\omega-\omega')t}). \tag{18}$$

Here, it is noted that we have $\phi_{+-}(t) \approx \phi_{+-}(\infty)$ and $\phi_{-+}(t) \approx \phi_{-+}(\infty)$ in the Markovian approximation ($\tau_R \to 0$). When we assume the resonant $\omega = \omega'$ or nearly resonant such that $|\omega - \omega'|/\tau_R \ll 1$, the functions $\phi_{-+}(t)$ and $\phi_{+-}(t)$ become

$$\phi_{-+}(t) = \kappa\tau_R(\bar{n}_{\mathrm{th}} + 1)\dot{f}(t) \qquad \phi_{+-}(t) = \kappa\tau_R\bar{n}_{\mathrm{th}}\dot{f}(t), \tag{19}$$

where $\dot{f}(t) = \mathrm{d}f(t)/\mathrm{d}t$ and the non-negative function $f(t)$ is given by

$$f(t) = \frac{t}{\tau_R} - 1 + \mathrm{e}^{-t/\tau_R} \tag{20}$$

and κ is a positive constant and $\bar{n}_{\mathrm{th}} = (\mathrm{e}^{\hbar\omega/k_{\mathrm{B}}T} - 1)^{-1}$. In deriving equation (19), we have used $\phi_{+-}(\infty) = \kappa\bar{n}_{\mathrm{th}}$ and $\phi_{-+}(\infty) = \kappa(\bar{n}_{\mathrm{th}} + 1)$. Then, in the limit of $\tau_R \to 0$, equation (3) reduces to the well-known Markovian quantum master equation of the Lindblad form.

When the functions $\phi_{-+}(t)$ and $\phi_{+-}(t)$ are given by equation (19), the Green function $G(t; z|t'; z')$ is simplified as

$$G_s(t; z|t'; z') = \frac{1}{\left(\bar{n}_{\mathrm{th}} + \frac{1+s}{2}\right)\left(1 - \gamma_{t-t'}^2\right)} \exp\left[-\frac{|z - z'\gamma_{t-t'}|^2}{\left(\bar{n}_{\mathrm{th}} + \frac{1+s}{2}\right)\left(1 - \gamma_{t-t'}^2\right)}\right] \tag{21}$$

with

$$\gamma_t = \mathrm{e}^{-\kappa\tau_R f(t)}. \tag{22}$$

For two-mode states, each mode of which is affected by the different thermal reservoir, the s-ordered quasi-probability distribution $F_s(t_1, t_2; z_1, z_2)$ is given by

$$F_s(t_1, t_2; z_1, z_2) = \int \frac{\mathrm{d}^2 z_1'}{\pi} \int \frac{\mathrm{d}^2 z_2'}{\pi} G_{1s}(t_1; z_1|t_1'; z_1') G_{2s}(t_2; z_2|t_2'; z_2') F_s(t_1', t_2'; z_1', z_2'). \tag{23}$$

The function $G_{ks}(t; z|t'; z')$ is obtained from equation (21) by replacing the parameters $\kappa, \tau_R, \bar{n}_{\mathrm{th}}$ with $\kappa_k, \tau_{kR}, \bar{n}_{k\mathrm{th}}$ ($k = 1, 2$). In equation (23), t_k is the transmission time of the kth mode through the non-Markovian quantum channel. The characteristic function which determines the statistical properties of a quantum state is the Fourier transformation of the Wigner function. Then the characteristic functions $C(t; \alpha)$ and $C(t_1, t_2; \alpha_1, \alpha_2)$ of single-mode and two-mode states in the non-Markovian quantum channel are calculated to be

$$C(t; \alpha) = \mathcal{G}(\alpha; t) C(0; \alpha\gamma_t) \tag{24}$$

and

$$C(t_1, t_2; \alpha_1, \alpha_2) = \mathcal{G}_1(\alpha_1; t_1)\mathcal{G}_2(\alpha_2; t_2) C(0, 0; \alpha_1\gamma_{1t}, \alpha_2\gamma_{2t}), \tag{25}$$

where the function $\mathcal{G}_k(\alpha; t)$ is given by

$$\mathcal{G}_k(\alpha; t) = \exp\left[-\left(\bar{n}_{k\mathrm{th}} + \tfrac{1}{2}\right)\left(1 - \gamma_{kt}^2\right)|\alpha|^2\right] \tag{26}$$

with $\gamma_{kt} = \mathrm{e}^{-\kappa_k\tau_{kR} f_k(t)}$. In equations (24) and (25), we have assumed that the initial time is zero.

3. Degradation of purity, distinguishability and non-classicality

In this section, we investigate degradation of purity, distinguishability and non-classicality of single-mode Gaussian states under the influence of the non-Markovian quantum channel defined by equations (11) and (21). In the Markovian limit ($\tau_R \to 0$), several authors have considered them in detail [35, 36]. A single-mode Gaussian state is completely characterized by three parameters $\alpha = \langle\hat{a}\rangle$, $\bar{n} = \langle\Delta\hat{a}^\dagger\Delta\hat{a}\rangle$ and $\bar{m} = -\langle(\Delta\hat{a})^2\rangle$ with $\Delta\hat{a} = \hat{a} - \langle\hat{a}\rangle$. The s-ordered quasi-probability distribution function $F_s(0; z)$ which represents a single-mode Gaussian state can be written in the form

$$F_s(0; z) = \sqrt{\det \mathsf{W}_s}\, \exp\left[-\tfrac{1}{2}(\mathsf{z} - \mathsf{a})^\dagger \mathsf{W}_s(\mathsf{z} - \mathsf{a})\right], \tag{27}$$

where $\mathsf{z} = (z^*, z)^\dagger$ and $\mathsf{a} = (\alpha^*, \alpha)^\dagger$ and the 2×2 Hermitian matrix W_s is

$$\mathsf{W}_s = \frac{1}{\left(\bar{n} + \frac{1+s}{2}\right)^2 - |\bar{m}|^2} \begin{pmatrix} \bar{n} + \frac{1+s}{2} & \bar{m} \\ \bar{m}^* & \bar{n} + \frac{1+s}{2} \end{pmatrix}. \tag{28}$$

The uncertainty relation requires the inequality $\bar{n}(\bar{n} + 1) \geqslant |\bar{m}|^2$, where the equality holds for pure Gaussian states. Furthermore, a single-mode Gaussian state is non-classical if and only if the inequality $\bar{n} < |\bar{m}|$ holds, where the analytic Glauber–Sudarshan P-function does not exist. Substituting equation (27) into equation (11) and applying equations (19) and (20), we can obtain the s-ordered quasi-probability distribution $F_s(t; z)$ in the non-Markovian quantum channel

$$F_s(t; z) = \sqrt{\det \mathsf{W}_s(t)}\, \exp\left(-\tfrac{1}{2}[\mathsf{z} - \mathsf{a}(t)]^\dagger \mathsf{W}_s(t)[\mathsf{z} - \mathsf{a}(t)]\right), \tag{29}$$

where $\mathsf{a}(t)$ and $\mathsf{W}_s(t)$ are obtained by replacing α, $\bar{n}$ and $\bar{m}$ in a and W_s with

$$\alpha(t) = \alpha\gamma_t \qquad \bar{n}(t) = \bar{n}_{\mathrm{th}} + (\bar{n} - \bar{n}_{\mathrm{th}})\gamma_t^2 \qquad \bar{m}(t) = \bar{m}\gamma_t^2. \tag{30}$$

Using the s-ordered quasi-probability distribution $F_s(t; z)$, we can investigate the decoherence of single-mode Gaussian states caused by the non-Markovian quantum channel.

3.1. Decay of purity of a single-mode Gaussian state

Purity of a quantum state $\hat{\rho}$ can be measured by means of the linear entropy $S_{\mathrm{L}}(\hat{\rho}) = 1 - \mathrm{Tr}\,\hat{\rho}^2$. For pure states, we have $S_{\mathrm{L}}(\hat{\rho}) = 0$. The linear entropy $S_{\mathrm{L}}(\hat{\rho}(t))$ of a single-mode Gaussian state in the non-Markovian quantum channel is given by

$$\begin{aligned} S_{\mathrm{L}}(\hat{\rho}(t)) &= 1 - \int \frac{\mathrm{d}^2 z}{\pi} F_{s=0}^2(t; z) \\ &= 1 - \frac{1}{\sqrt{[2\bar{n}(t) + 1]^2 - 4|\bar{m}(t)|^2}}. \end{aligned} \tag{31}$$

In particular, when the initial Gaussian state is pure, where the equality $\bar{n}(\bar{n} + 1) = |\bar{m}|^2$ holds, the linear entropy $S_{\mathrm{L}}(\hat{\rho}(t))$ becomes

$$S_{\mathrm{L}}(\hat{\rho}(t)) = 1 - \frac{1}{\sqrt{1 + 4\left(1 - \gamma_t^2\right)\left[\bar{n}_{\mathrm{th}}(\bar{n}_{\mathrm{th}} + 1)\left(1 + \gamma_t^2\right) + (2\bar{n}_{\mathrm{th}} + 1)(\bar{n} - \bar{n}_{\mathrm{th}})\gamma_t^2\right]}} \tag{32}$$

which is plotted in figure 1. The figure clearly shows that the non-Markovian effect of the thermal reservoir suppresses the degradation of purity of single-mode Gaussian states.

3.2. Decay of distinguishability of a single-mode Gaussian state

One of the distinguishability measures between two quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ is the fidelity $F(\hat{\rho}_1, \hat{\rho}_2)$ given by

$$F(\hat{\rho}_1, \hat{\rho}_2) = \left[\mathrm{Tr}\left(\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}\right)^{1/2}\right]^2. \tag{33}$$

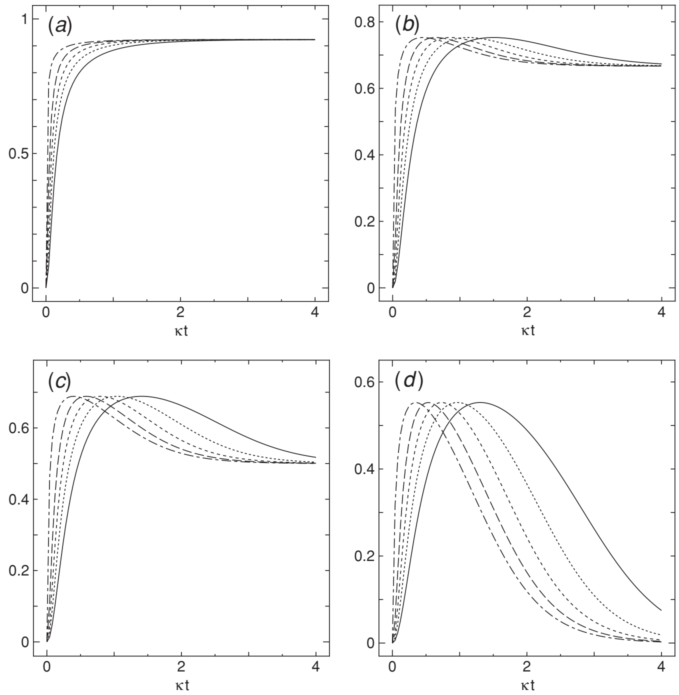

Figure 1. The linear entropy of the single-mode Gaussian state with $\bar{n} = 4.0$ under the influence of the non-Markovian quantum channel, where (*a*) $\bar{n}_{\mathrm{th}} = 6.0$, (*b*) $\bar{n}_{\mathrm{th}} = 1.0$, (*c*) $\bar{n}_{\mathrm{th}} = 0.5$ and (*d*) $\bar{n}_{\mathrm{th}} = 0.0$. In the figures, the solid line represents $\kappa\tau_R = 2.0$, the dotted line $\kappa\tau_R = 1.0$, the short-dashed line $\kappa\tau_R = 0.5$, the dashed line $\kappa\tau_R = 0.2$ and the dash-dotted line $\kappa\tau_R = 0.0$. The last corresponds to the Markovian limit.

For two Gaussian states characterized by the parameters α_k, $\bar{n}_k$ and $\bar{m}_k$ $(k = 1, 2)$, the fidelity $F(\hat{\rho}_1, \hat{\rho}_2)$ can be analytically calculated [43]. In particular, when $\alpha_1 = \alpha_2$, we obtain

$$F(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{\sqrt{X+Y} - \sqrt{X}} \tag{34}$$

with

$$X = 2[\bar{n}_1(\bar{n}_1 + 1) - |\bar{m}_1|^2][\bar{n}_2(\bar{n}_2 + 1) - |\bar{m}_2|^2] \tag{35}$$

$$Y = (\bar{n}_1 + \bar{n}_2 + 1)^2 - |\bar{m}_1 + \bar{m}_2|^2. \tag{36}$$

When we assume that $\bar{n}_1 = \bar{n}_2 = \bar{n}$ and $\bar{m}_1 = -\bar{m}_2 = \sqrt{\bar{n}(\bar{n}+1)}$ for the initial Gaussian states, the fidelity between the two Gaussian states in the non-Markovian quantum channel is obtained by substituting $\bar{n}_k \to \bar{n}_{\mathrm{th}} + (\bar{n} - \bar{n}_{\mathrm{th}})\gamma_t^2$ $(k = 1, 2)$, $\bar{m}_1 \to \sqrt{\bar{n}(\bar{n}+1)}\gamma_t^2$ and $\bar{m}_2 \to -\sqrt{\bar{n}(\bar{n}+1)}\gamma_t^2$ in equations (34)–(36). In this case, the fidelity is plotted in figure 2. It is found from the figure that decay of the distinguishability between single-mode Gaussian states is made smaller by the non-Markovian effect of the thermal reservoir.

3.3. Decay of non-classicality of a single-mode Gaussian state

To measure the non-classicality of a quantum state, we can use the non-classical depth τ_c which is defined as the minimum value of the thermal noise necessary for the Glauber–Sudarshan

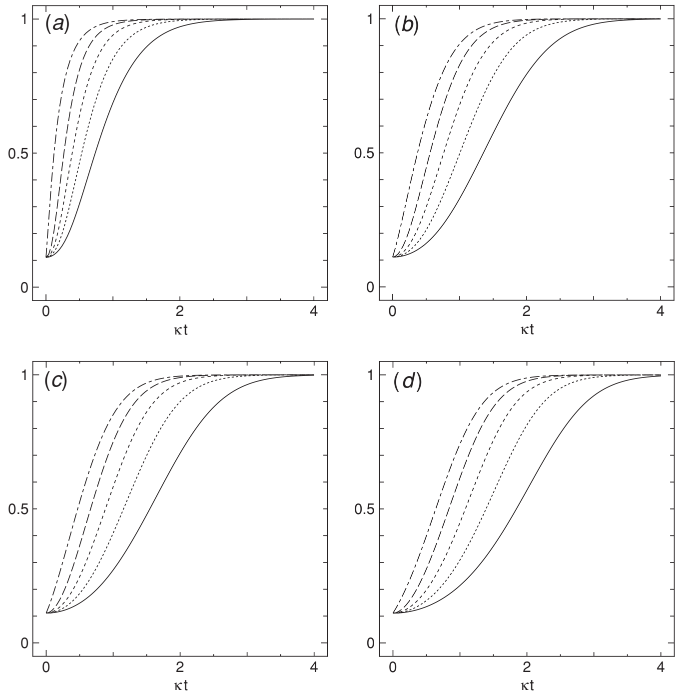

Figure 2. The fidelity between the two Gaussian states with $\bar{n} = 4.0$ under the influence of the non-Markovian quantum channel, where (*a*) $\bar{n}_{\text{th}} = 6.0$, (*b*) $\bar{n}_{\text{th}} = 1.0$, (*c*) $\bar{n}_{\text{th}} = 0.5$ and (*d*) $\bar{n}_{\text{th}} = 0.0$. In the figures, the solid line represents $\kappa\tau_R = 2.0$, the dotted line $\kappa\tau_R = 1.0$, the short-dashed line $\kappa\tau_R = 0.5$, the dashed line $\kappa\tau_R = 0.2$ and the dash-dotted line $\kappa\tau_R = 0.0$. The last corresponds to the Markovian limit.

P-function to be non-negative and less singular than δ-function [44]. Let $P(z) = F_{s=-1}(0; z)$ be the Glauber–Sudarshan P-function of a quantum state $\hat{\rho}$ and we introduce the function $R(z)$ by

$$R(z) = \frac{1}{\tau}\int \frac{\mathrm{d}^2 z'}{\pi}\, \mathrm{e}^{-|z-z'|^2/\tau} P(z'). \tag{37}$$

Then the non-classical depth τ_c of the quantum state $\hat{\rho}$ is given by the minimum value of τ such that the function $R(z)$ is non-negative and less singular than the δ-function. It is easy to see that the parameter s which determines the operator ordering is related to τ by the relation

$$2\tau = 1 + s. \tag{38}$$

For single-mode Gaussian states, the s-ordered quasi-probability distribution function becomes non-singular if the inequality $\bar{n} + (1+s)/2 \geqslant |\bar{m}|$ holds. Hence, the non-classical depth τ_c of a single-mode Gaussian state is given by

$$\tau_c = \max[|\bar{m}| - \bar{n}, 0]. \tag{39}$$

For the initial Gaussian state with the non-classical depth $\tau_c(0)$, we obtain the non-classical depth $\tau_c(t)$ of the Gaussian state at time t under the influence of the non-Markovian quantum channel

$$\begin{aligned}\tau_c(t) &= \max[|\bar{m}(t)| - \bar{n}(t), 0] \\ &= \max\left[\tau_c(0)\gamma_t^2 - \bar{n}_{\text{th}}\left(1-\gamma_t^2\right), 0\right].\end{aligned} \tag{40}$$

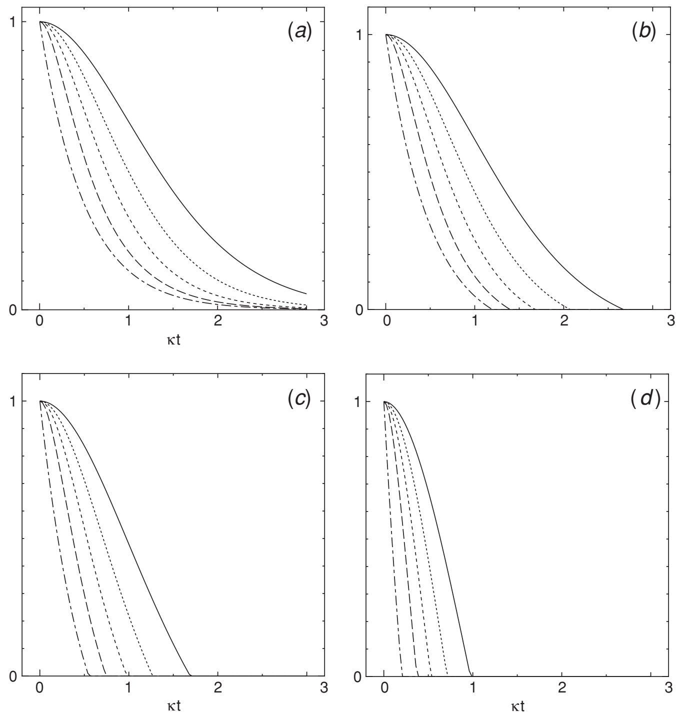

Figure 3. The non-classical depth of the single-mode Gaussian state with $\tau_c(0) = 1.0$ under the influence of the non-Markovian quantum channel, where (*a*) $\bar{n}_{\rm th} = 0.0$, (*b*) $\bar{n}_{\rm th} = 0.1$, (*c*) $\bar{n}_{\rm th} = 0.5$ and (*d*) $\bar{n}_{\rm th} = 2.0$. In the figures, the solid line represents $\kappa\tau_R = 2.0$, the dotted line $\kappa\tau_R = 1.0$, the short-dashed line $\kappa\tau_R = 0.5$, the dashed line $\kappa\tau_R = 0.2$ and the dash-dotted line $\kappa\tau_R = 0.0$. The last corresponds to the Markovian limit.

This result shows that if the temperature of the thermal reservoir is zero and the initial Gaussian state has the non-zero non-classical depth, the quantum state is always non-classical in the non-Markovian quantum channel. The non-classical depth of single-mode Gaussian states under the influence of the non-Markovian quantum channel is plotted in figure 3. The non-Markovian effect of the thermal reservoir makes longer the survival time of the non-classicality of a single-mode Gaussian state. The results obtained in this section show that the non-Markovian effect suppresses the decoherence of purity, distinguishability and non-classicality of single-mode Gaussian states.

4. Decoherence of entanglement of two-mode Gaussian states

In this section, calculating the entanglement of formation, we investigate the decoherence of entanglement of a two-mode Gaussian state under the influence of the non-Markovian quantum channel. Since the average values $\langle\hat{a}_k\rangle$ $(k = 1, 2)$ of the annihilation operator of the each mode does not affect the entanglement, we can assume $\langle\hat{a}_k\rangle = 0$ without any loss of generality. Then a two-mode Gaussian state is, in general, characterized by parameters $\bar{n}_k = \langle\hat{a}_k^\dagger\hat{a}_k\rangle$, $\bar{m}_k = -\langle\hat{a}_k^2\rangle$, $\bar{m}_c = -\langle\hat{a}_1\hat{a}_2\rangle$ and $\bar{m}_s = \langle\hat{a}_1\hat{a}_2^\dagger\rangle$ $(k = 1, 2)$. In particular, we pay special attention to the case of $\bar{m}_c = \bar{m}$ and $\bar{m}_k = 0$ $(k = 1, 2, s)$. Such a two-mode Gaussian state is called the mixed EPR state [45] including the noisy two-mode squeezed vacuum state which is the most important resource in continuous variable quantum information processing.

The mixed EPR state is inseparable or entangled if and only if the inequality $\bar{n}_1\bar{n}_2 < |\bar{m}|^2$ holds. The inequality $\bar{n}_k(\bar{n}_k + 1) \geqslant |\bar{m}|^2$ $(k = 1, 2)$ is always satisfied due to the uncertainty relation. Using equation (23) or (25), we can obtain the time-evolution of the parameters $\bar{n}_k$ and $\bar{m}$ under the influence of the non-Markovian quantum channel

$$\bar{n}_k(t) = \bar{n}_{\mathrm{th}} + (\bar{n}_k - \bar{n}_{\mathrm{th}})\gamma_t^2 \qquad \bar{m}(t) = \bar{m}\gamma_t^2. \tag{41}$$

To obtain the entanglement of formation of the mixed EPR state, we need the covariance matrix, the element of which is $\mathsf{M}_{jk} = (1/2)[\langle\hat{x}_j\hat{x}_k\rangle + \langle\hat{x}_k\hat{x}_j\rangle]$, where $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) = (\hat{q}_1, \hat{p}, \hat{q}_2, \hat{p}_2)$, and $\hat{q}_k$ and $\hat{p}_k$ are canonical position and momentum operators of the kth mode, respectively, satisfying the canonical commutation relation $[\hat{q}_j, \hat{p}_k] = \mathrm{i}\hbar\delta_{jk}$. For the mixed EPR state, we obtain the covariance matrix

$$\mathsf{M} = \begin{pmatrix} \bar{n}_1 + \frac{1}{2} & 0 & -\mathrm{Re}\,\bar{m} & -\mathrm{Im}\,\bar{m} \\ 0 & \bar{n}_1 + \frac{1}{2} & -\mathrm{Im}\,\bar{m} & \mathrm{Re}\,\bar{m} \\ -\mathrm{Re}\,\bar{m} & -\mathrm{Im}\,\bar{m} & \bar{n}_2 + \frac{1}{2} & 0 \\ -\mathrm{Im}\,\bar{m} & \mathrm{Re}\,\bar{m} & 0 & \bar{n}_2 + \frac{1}{2} \end{pmatrix}. \tag{42}$$

The standard form of the covariance matrix is given by

$$\mathsf{M}_s = \begin{pmatrix} \bar{n}_1 + \frac{1}{2} & 0 & |\bar{m}| & 0 \\ 0 & \bar{n}_1 + \frac{1}{2} & 0 & -|\bar{m}| \\ |\bar{m}| & 0 & \bar{n}_2 + \frac{1}{2} & 0 \\ 0 & |\bar{m}| & 0 & \bar{n}_2 + \frac{1}{2} \end{pmatrix} \tag{43}$$

which is derived by the canonical transformation from the matrix M. It is found from equation (25) that the matrix M_s of the mixed EPR state in the non-Markovian quantum channel is obtained by replacing in equation (43) the parameters $\bar{n}_k$ and $\bar{m}$ with $\bar{n}_k(t)$ and $\bar{m}(t)$, respectively, given by equation (41). When the equality $\bar{n}_1 = \bar{n}_2$ holds, the mixed EPR state is called symmetric. In the rest of this section, we consider the decoherence of the entanglement of the symmetric mixed EPR state.

Since the symmetric mixed EPR state has the standard form of the covariance matrix (43) with $\bar{n}_1 = \bar{n}_2$ $(\equiv \bar{n})$, under the influence of the non-Markovian quantum channel, the entanglement of formation [46] is given by

$$\begin{aligned} E_F(t) = -1 &+ \frac{1}{2}\left[\frac{1}{2}\left(\lambda(t) + \frac{1}{\lambda(t)}\right) + 1\right]\log_2\left[\frac{1}{2}\left(\lambda(t) + \frac{1}{\lambda(t)}\right) + 1\right] \\ &- \frac{1}{2}\left[\frac{1}{2}\left(\lambda(t) + \frac{1}{\lambda(t)}\right) - 1\right]\log_2\left[\frac{1}{2}\left(\lambda(t) + \frac{1}{\lambda(t)}\right) - 1\right], \end{aligned} \tag{44}$$

where the parameter $\lambda(t)$ is defined by

$$\lambda(t) = \min[2\bar{n}(t) - 2|\bar{m}(t)| + 1, 1]. \tag{45}$$

We can derive the inequality $0 < \lambda(t) \leqslant 1$ from the uncertainty relation. The mixed EPR state becomes separable if and only if the equality $\lambda(t) = 1$ holds which is equivalent to the inequality $\bar{n}(t) \geqslant |\bar{m}(t)|$. Substituting equation (41) into equation (45), we obtain

$$\lambda(t) = \min\left[2\left(\bar{n}_{\mathrm{th}} + (\bar{n} - \bar{n}_{\mathrm{th}})\gamma_t^2\right) - 2|\bar{m}|^2\gamma_t^2 + 1, 1\right]. \tag{46}$$

In particular, when the temperature of the thermal reservoir is zero, that is $\bar{n}_{\mathrm{th}} = 0$, we obtain $\lambda(t) = \min\left[2(\bar{n} - \bar{m})\gamma_t^2 + 1, 1\right]$. Since the inequality $\sqrt{\bar{n}(\bar{n}+1)} \geqslant |\bar{m}| > \bar{n}$ is satisfied for the initially entangled state, the inequality $1 > \lambda(t) > 0$ is always established. This implies that the mixed EPR state which is initially entangled remains entangled under the influence of the non-Markovian thermal reservoir with $\bar{n}_{\mathrm{th}} = 0$. The entanglement of formation is plotted in figure 4. The figure shows that the non-Markovian effect of the thermal reservoir suppresses the decoherence of entanglement of the symmetric mixed EPR state.

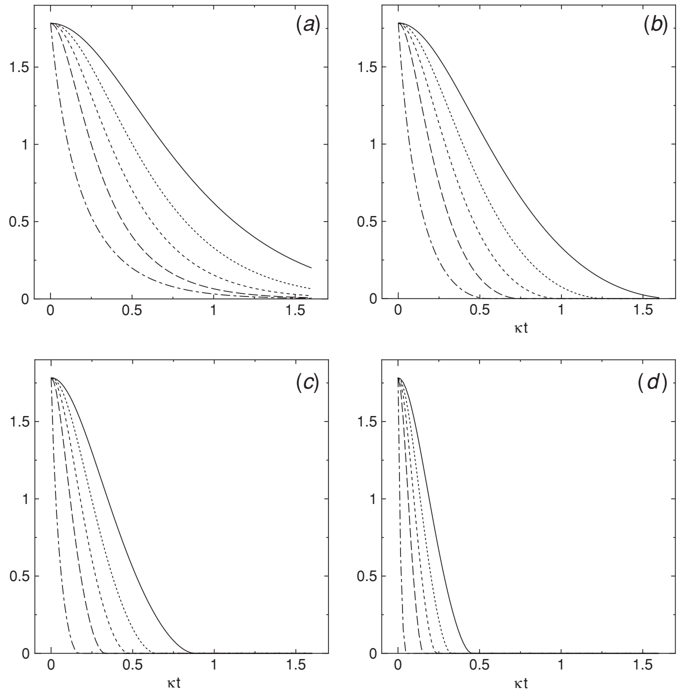

Figure 4. The entanglement of formation of the symmetric mixed EPR state with $\bar{n} = 3.0$ and $\bar{m} = 3.4$ under the influence of the non-Markovian quantum channel, where (*a*) $\bar{n}_{\text{th}} = 0.0$, (*b*) $\bar{n}_{\text{th}} = 0.2$, (*c*) $\bar{n}_{\text{th}} = 1.0$ and (*d*) $\bar{n}_{\text{th}} = 4.0$. In the figures, the solid line represents $\kappa\tau_R = 2.0$, the dotted line $\kappa\tau_R = 1.0$, the short-dashed line $\kappa\tau_R = 0.5$, the dashed line $\kappa\tau_R = 0.2$ and the dash-dotted line $\kappa\tau_R = 0.0$. The last corresponds to the Markovian limit.

5. Entanglement-breaking channel by the non-Markovian thermal reservoir

In this section, we obtain the condition that the non-Markovian quantum channel defined by equations (11) and (21) becomes an entanglement-breaking channel. A quantum channel $\hat{\mathcal{L}}$ is called entanglement-breaking if $(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}$ is separable for any two-mode state $\hat{W}$ [47–49]. The condition for $\hat{\mathcal{L}}$ to be an entanglement-breaking channel is that for any single-mode state $\hat{\rho}$, there are quantum states $\hat{\rho}_k$ and positive operator-valued measure $\hat{M}_k$ such that [47, 48]

$$\hat{\mathcal{L}}\hat{\rho} = \sum_k \hat{\rho}_k \, \text{Tr}[\hat{M}_k \hat{\rho}]. \tag{47}$$

In the following, using the method developed in [50–52], we investigate whether the non-Markovian quantum channel satisfies the continuous version of condition (47) [49].

From equations (11) and (21), the input–output relation of the Wigner function in the non-Markovian quantum channel is given by

$$W(z;t) = \int \frac{\mathrm{d}^2 z'}{\pi} \frac{1}{\Delta_t} \, \mathrm{e}^{-|z - z'\gamma_t|^2/\Delta_t} W(z';0) \tag{48}$$

with

$$\Delta_t = \left(\bar{n}_{\text{th}} + \tfrac{1}{2}\right)\left(1 - \gamma_t^2\right). \tag{49}$$

Using the property of the Gaussian integration, we can formally rewrite equation (48) as

$$
\begin{aligned}
W(z;t) &= \int \frac{\mathrm{d}^2 z'}{\pi} \left[\frac{2}{\gamma_t^2\left(\Delta_t - \frac{1}{2}\gamma_t^2\right)} \int \frac{\mathrm{d}^2 z''}{\pi}\, \mathrm{e}^{-|z-z''|^2/(\Delta_t - \frac{1}{2}\gamma_t^2)}\, \mathrm{e}^{-2|z''-z'\gamma_t|^2/\gamma_t^2} \right] W(z';0) \\
&= \frac{1}{\Delta_t - \frac{1}{2}\gamma_t^2} \int \frac{\mathrm{d}^2 z''}{\pi}\, \mathrm{e}^{-|z-z''|^2/(\Delta_t - \frac{1}{2}\gamma_t^2)} \times 2 \int \frac{\mathrm{d}^2 z'}{\pi}\, \mathrm{e}^{-2|z''-z'|^2} W(z';0) \\
&= \int \frac{\mathrm{d}^2 z''}{\pi} \left[\frac{1}{\mathcal{N}_t + \frac{1}{2}} \mathrm{e}^{-|z-z''|^2/(\mathcal{N}_t + \frac{1}{2})} \right] Q(z'';0),
\end{aligned}
\tag{50}
$$

where $Q(z;0)$ is the Husimi Q-function of the initial quantum state $\hat{\rho}$ and the parameter $\mathcal{N}_t$ is given by

$$
\mathcal{N}(t) = \Delta_t - \frac{1}{2}\gamma_t^2 - \frac{1}{2} = \bar{n}_{\mathrm{th}}\left(1-\gamma_t^2\right) - \gamma_t^2. \tag{51}
$$

If the parameter $\mathcal{N}_t$ is non-negative, we can define the thermal state $\hat{\rho}_{\mathrm{th}}$ by the relation

$$
\hat{\rho}_{\mathrm{th}} = \frac{1}{\mathcal{N}_t} \int \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{-|z|^2/\mathcal{N}_t} |z\rangle\langle z| \tag{52}
$$

the Wigner function of which is equal to the Gaussian distribution in $[\cdots]$ of equation (50). Here $|z\rangle$ is a coherent state.

Therefore, we find that the output state $\hat{\rho}(t)$ from the non-Markovian quantum channel is expressed as

$$
\hat{\rho}(t) = \int \frac{\mathrm{d}^2 z}{\pi} \hat{D}(z\gamma_t)\hat{\rho}_{\mathrm{th}}\hat{D}^\dagger(z\gamma_t) Q(z;0), \tag{53}
$$

where $\hat{D}(z)$ is the displacement operator. Here, we define quantum state $\hat{\rho}_z$ and positive operator-valued measure $\hat{M}_z$ by

$$
\hat{\rho}_z = \hat{D}(z\gamma_t)\hat{\rho}_{\mathrm{th}}\hat{D}^\dagger(z\gamma_t) \qquad \hat{M}_z = \frac{1}{\pi}|z\rangle\langle z| \tag{54}
$$

and we denote the non-Markovian quantum channel as $\hat{\mathcal{L}}_t$. Then we can obtain the expression of the non-Markovian quantum channel

$$
\hat{\mathcal{L}}_t \hat{\rho} = \int \frac{\mathrm{d}^2 z}{\pi} \hat{\rho}_z \operatorname{Tr}[\hat{M}_z \hat{\rho}] \tag{55}
$$

which is equal to the continuous version of condition (47). This result implies that the non-Markovian quantum channel is entanglement breaking. It is important to note that this result is valid if the parameter $\mathcal{N}_t$ is non-negative, namely, $\mathcal{N}_t \geqslant 0$. From equation (51), the non-negativity of the parameter $\mathcal{N}_t$ is equivalent to the inequality

$$
\frac{\bar{n}_{\mathrm{th}}}{\bar{n}_{\mathrm{th}} + 1} \geqslant \gamma_t^2 \tag{56}
$$

which is the condition that the non-Markovian quantum channel becomes entanglement breaking. We find from equation (56) that if the temperature of the thermal reservoir is zero ($\bar{n}_{\mathrm{th}} = 0$), the non-Markovian quantum channel does not become entanglement breaking while if the thermal reservoir has the infinitely high temperature ($\bar{n}_{\mathrm{th}} \gg 1$), it is always entanglement breaking. The time τ_B (the entanglement-breaking time) at which the non-Markovian quantum channel becomes entanglement breaking is plotted as the function of the reservoir correlation time τ_R in figure 5. It is easy to see from the figure that the entanglement-breaking time is made longer by the non-Markovian effect of the thermal reservoir.

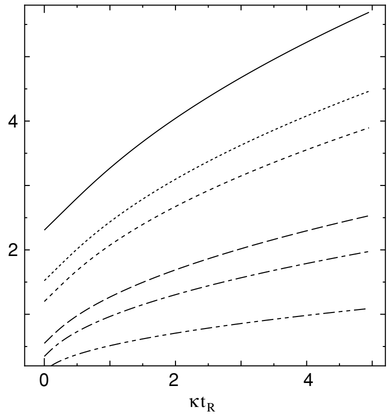

Figure 5. The entanglement-breaking time $\kappa\tau_B$ of the non-Markovian quantum channel, where the solid line represents $\bar{n}_{\text{th}} = 0.01$, the dotted line $\bar{n}_{\text{th}} = 0.05$, the short-dashed line $\bar{n}_{\text{th}} = 0.1$, the dashed line $\bar{n}_{\text{th}} = 0.5$, dash-dotted line $\bar{n}_{\text{th}} = 1.0$ and dash-dot-dotted line $\bar{n}_{\text{th}} = 4.0$.

6. Continuous variable quantum teleportation

In this section, we consider the continuous variable quantum teleportation [29] when a sender and a receiver share the mixed EPR state through the non-Markovian quantum channel. In this case, the Wigner function $W(z_1, z_2; t)$ of the quantum state shared by the sender and the receiver is given by

$$W(z_1, z_2) = \frac{1}{\mathcal{G}(\bar{n}_1, \bar{n}_2, \bar{m}; t)} \exp\left[-\frac{\Phi(z_1, z_2; \bar{n}_1, \bar{n}_2, \bar{m}; t)}{\mathcal{G}(\bar{n}_1, \bar{n}_2, \bar{m}; t)}\right] \tag{57}$$

with

$$\Phi(z_1, z_2; \bar{n}_1, \bar{n}_2, \bar{m}; t) = \left[\bar{n}_2(t) + \tfrac{1}{2}\right]|z_1|^2 + \left[\bar{n}_1(t) + \tfrac{1}{2}\right]|z_2|^2 + \bar{m}^*(t) z_1 z_2 + \bar{m}(t) z_1^* z_2^* \tag{58}$$

and

$$\mathcal{G}(\bar{n}_1, \bar{n}_2, \bar{m}; t) = \left[\bar{n}_1(t) + \tfrac{1}{2}\right]\left[\bar{n}_2(t) + \tfrac{1}{2}\right] - |\bar{m}(t)|^2 \tag{59}$$

where $\bar{n}_k(t)$ and $\bar{m}(t)$ are given by equation (41). To do the continuous variable quantum teleportation, the sender who is provided an unknown quantum state $\hat{\rho}_{\text{in}}$ performs the simultaneous measurement of position and momentum and informs the receiver of the measurement outcome via a classical communication channel. The receiver who is known the measurement outcome applies the displacement operator to her/his mode of the shared quantum state, where the magnitude of the displacement depends on the measurement outcome. Then the receiver obtains the quantum state $\hat{\rho}_{\text{out}}$ in average [53, 54]

$$\hat{\rho}_{\text{out}} = \int \frac{\mathrm{d}^2\alpha}{\pi} G(\alpha) \hat{D}(\alpha) \hat{\rho}_{\text{in}} \hat{D}^\dagger(\alpha), \tag{60}$$

where the function $G(\alpha)$ is given by

$$\begin{aligned} G(\alpha) &= \int \frac{\mathrm{d}^2 z}{\pi} W(z^* - \alpha^*, \alpha) \\ &= \frac{1}{\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t)} \exp\left[-\frac{|\alpha|^2}{\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t)}\right] \end{aligned} \tag{61}$$

with

$$\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t) = \bar{n}_1(t) + \bar{n}_2(t) + \bar{m}^*(t) + \bar{m}(t) + 1 \tag{62}$$

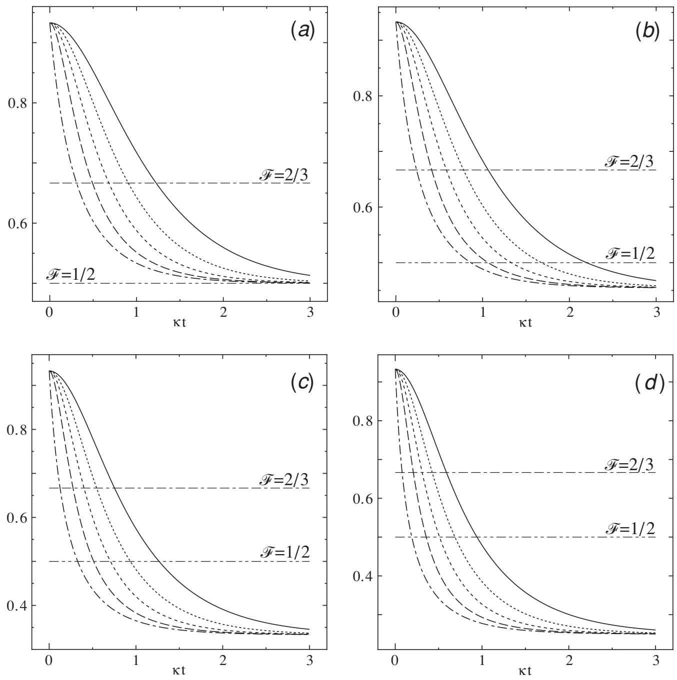

Figure 6. The fidelity of the coherent state obtained by the quantum teleportation by means of the symmetric mixed EPR state with $\bar{n} = 3.0$ and $\bar{m} = -2\sqrt{3}$ under the influence of the non-Markovian quantum channel, where (*a*) $\bar{n}_{\rm th} = 0.0$, (*b*) $\bar{n}_{\rm th} = 0.1$, (*c*) $\bar{n}_{\rm th} = 0.5$ and (*d*) $\bar{n}_{\rm th} = 1.0$ In the figures, the solid line represents $\kappa\tau_R = 2.0$, the dotted line $\kappa\tau_R = 1.0$, the short-dashed line $\kappa\tau_R = 0.5$, the dashed line $\kappa\tau_R = 0.2$ and the dash-dotted line $\kappa\tau_R = 0.0$. The last corresponds to the Markovian limit. $\mathcal{F} = 1/2$ is the upper bound in the classical teleportation. The inequality $\mathcal{F} > 1/2$ implies the quantumness of the teleportation. $\mathcal{F} = 2/3$ is the lower bound of the non-cloning limit [55].

the non-negativity of which is ensured by the uncertainty relation. It is easy to see from equations (60) and (61) that the performance of the continuous variable quantum teleportation becomes higher as the parameter $\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t)$ is smaller. In the limit of $\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t) \to 0$, the complete quantum teleportation in which the teleported state is identical with the original state is possible. For example, when a coherent state is teleported, the fidelity $\mathcal{F}$ between the original and teleported states is given by

$$\mathcal{F} = \frac{1}{1 + \Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t)}. \tag{63}$$

Substituting equation (41) into equation (62), we obtain the time-dependence of the parameter $\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t)$,

$$\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t) = (\bar{n}_{\rm th} + 1)\left(1 - \gamma_t^2\right) + \Delta(\bar{n}_1, \bar{n}_2, \bar{m}; 0)\gamma_t^2. \tag{64}$$

When the non-Markovian quantum channel is entanglement breaking, the inequality (56) is satisfied. In this case, we can derive the inequality

$$\Delta(\bar{n}_1, \bar{n}_2, \bar{m}; t) \geqslant \frac{2\bar{n}_{\rm th} + 1}{\bar{n}_{\rm th} + 1} \geqslant 1 \tag{65}$$

which provides the inequality $\mathcal{F} \leqslant 1/2$. This means that when the quantum channel used for sharing the mixed EPR state is entanglement breaking, the non-classical property of the quantum state cannot be teleported [54]. To show the non-Markovian effect on the continuous variable quantum teleportation, the fidelity given by equation (63) is plotted in figure 6. The non-Markovian effect makes better the performance of the continuous variable quantum teleportation.

7. Concluding remarks

In this paper, we have considered the non-Markovian effect of the thermal reservoir on the quantum information carried by single-mode and two-mode Gaussian states. We have obtained the non-Markovian quantum channel by means of the projection operator method that yields the time-convolutionless quantum master equation. We have investigated the degradation of purity, distinguishability and non-classicality for single-mode Gaussian states and the decoherence of entanglement of two-mode Gaussian states under the influence of the non-Markovian thermal reservoir. The results show that the non-Markovian effect suppresses the decoherence of the quantum states. We have also obtained the condition that the non-Markovian quantum channel becomes an entanglement-breaking channel. Furthermore, we have investigated the performance of the continuous variable quantum teleportation by means of the mixed EPR state shared through the non-Markovian quantum channel.

Acknowledgments

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Appendix A. Derivation of equations (11)–(14)

To solve the differential equation (10) corresponding to the non-Markovian quantum master equation, we introduce the function $\tilde{F}_s(t;z)$ by the relation

$$\tilde{F}_s(t;z) = \mathrm{e}^{-\Gamma(t,t')\left(\frac{\partial}{\partial z}z+\frac{\partial}{\partial z^*}z^*\right)} F_s(t;z), \tag{A.1}$$

where the parameter $\Gamma(t,t')$ is given by equation (13). Using the commutation relation,

$$[\partial^2/\partial z\partial z^*, z\partial/\partial z + z^*\partial/\partial z^*] = 2\partial^2/\partial z\partial z^* \tag{A.2}$$

we can find that the function $\tilde{F}_s(t;z)$ obeys the differential equation

$$\frac{\partial}{\partial t}\tilde{F}_s(t;z) = [(1+s)\phi_{-+}(t) + (1-s)\phi_{+-}(t)]\,\mathrm{e}^{2\Gamma(t,t')}\left(\frac{\partial^2}{\partial z\partial z^*}\right)\tilde{F}_s(t;z) \tag{A.3}$$

which yields the solution

$$\tilde{F}_s(t;z) = \mathrm{e}^{\Upsilon_s(t,t')\frac{\partial^2}{\partial z\partial z^*}}\tilde{F}_s(t';z') \tag{A.4}$$

with

$$\Upsilon_s(t,t') = \int_{t'}^{t}\mathrm{d}\tau[(1+s)\phi_{-+}(\tau) + (1-s)\phi_{+-}(\tau)]\,\mathrm{e}^{2\Gamma(\tau,t')}. \tag{A.5}$$

Thus the quasi-probability distribution $F_s(t;z)$ is given by

$$F_s(t;z) = \mathrm{e}^{\Gamma(t,t')\left(\frac{\partial}{\partial z}z+\frac{\partial}{\partial z^*}z^*\right)}\,\mathrm{e}^{\Upsilon_s(t,t')\frac{\partial^2}{\partial z\partial z^*}}F_s(t';z'). \tag{A.6}$$

The Green function $G(t;z|t';z')$ that satisfies equation (11) can be defined by

$$G(t;z|t';z') = \mathrm{e}^{\Gamma(t,t')\left(\frac{\partial}{\partial z}z+\frac{\partial}{\partial z^*}z^*\right)}\,\mathrm{e}^{\Upsilon_s(t,t')\frac{\partial^2}{\partial z\partial z^*}}\pi\delta^{(2)}(z-z'). \tag{A.7}$$

Using the Fourier representation of the δ-function and the relation $\mathrm{e}^{az\partial/\partial z} f(z) = f(\mathrm{e}^{a} z)$, we can obtain the Green function

$$G(t; z|t'; z') = \mathrm{e}^{\Gamma(t,t')(\frac{\partial}{\partial z}z + \frac{\partial}{\partial z^*}z^*)}\, \mathrm{e}^{\Upsilon_s(t,t')\frac{\partial^2}{\partial z \partial z^*}} \int \frac{\mathrm{d}^2\alpha}{\pi}\, \mathrm{e}^{\alpha^*(z-z') - \alpha(z-z')^*}$$
$$= \mathrm{e}^{2\Gamma(t,t')} \int \frac{\mathrm{d}^2\alpha}{\pi}\, \mathrm{e}^{-\Upsilon(t,t')|\alpha|^2}\, \mathrm{e}^{\alpha^*(z\mathrm{e}^{\Gamma(t,t')} - z') - \alpha(z\mathrm{e}^{\Gamma(t,t')} - z')^*}$$
$$= \frac{1}{\Delta_s(t,t')} \exp\left[-\frac{|z - z'\,\mathrm{e}^{-\Gamma(t,t')}|^2}{\Delta_s(t,t')} \right]. \tag{A.8}$$

where the parameter $\Delta_s(t, t')$ is given by equation (14). Therefore, we have derived equations (11)–(14).

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